A&A manuscript no.
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07.(02.08.1;02.20.1;08.09.3)

ASTRONOMY AND ASTROPHYSICS

The small-Péclet-number approximation in stellar radiative zones

F. Lignières

Département de Recherche Spatiale et Unité de Recherche associée au CNRS 264, Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France (internet:Francois.Lignieres@obspm.fr)

Received 15 September 1998 / Accepted 21 May 1999

Abstract. We present an asymptotic form of the Boussinesq equations in the limit of small Péclet numbers i.e. when the time scale of motions is much larger than the time scale of thermal diffusion. We find that, in this limit, the effects of thermal diffusion and stable stratification combine in a single physical process. This process is an anisotropic dissipation (not effective for horizontal motions) which acts primarily on large scale motions. The small-Péclet-number approximation presents also the great practical interest to avoid the numerical difficulty induced by the huge separation between the diffusive and dynamical time scales. The relevance of this approximation to study the flow dynamics within the stellar radiative zones is considered.

Key words: Hydrodynamics – turbulence – Stars: interiors

1. Introduction

The understanding of the flow dynamics within stellar radiative zones constitutes a major challenge for the current theory of stellar evolution. These motions transport chemical elements and it turns out that their contribution might reconcile the existing models of stellar structure with the observations of the surface abundances (Pinsonneault 1998). By transporting angular momentum, such flows also play an important role in the evolution of star's rotation. In particular, they could explain the nearly solid body rotation of the solar radiative zone which has been revealed by helioseismology (Gough et al. 1996).

We consider here the effects of the very high thermal diffusivity of stellar interiors on the dynamics of these motions. In most cases, radiation dominates the thermal exchanges within radiative zones. This heat transport is so efficient that the thermal diffusivities associated with the radiative flux are larger by several orders of magnitude than the thermal diffusivities encountered in colder media like planetary atmospheres. For example, the thermal diffusivity varies between 10^5 and 10^7 cm^{2}s^{-1} inside the sun whereas it is equal to 0.18 cm^{2}s^{-1} in the standard conditions of the terrestrial atmosphere.

This property of the stellar fluid is expected to strongly affect the flow dynamics especially inside the stably stratified radiative zone where the time scale of thermal diffusion appears to be shorter than the dynamical time scale characterizing radial motions. Helioseismology data show that the thermal structure of this region is very close to the one predicted by hydrostatic models, indicating that existing fluid motions are not fast enough to modify significantly the thermal structure built up by the radiative flux (Canuto & Christensen-Dalsgaard 1998).

Qualitatively, the damping of temperature fluctuations by thermal diffusion is expected to have two main effects on the dynamics. The first one is to reduce the amplitude of the buoyancy force. This restoring force acts on fluid parcels displaced from their equilibrium level and is proportional to the density difference between the parcel and its environment. Since density fluctuations are proportional to temperature fluctuations for incompressible motions, fast thermal exchanges reduce the force amplitude. An important consequence of this effect is to favour the onset of shear layer instabilities in stably stratified layers (Dudis 1974, Zahn 1974).

The second main effect of the thermal diffusion is to increase the dissipation of kinetic energy. Any vertical motion in a quiescent atmosphere induces a work of the buoyancy force so that a fraction of the injected kinetic energy is necessarily transformed into potential energy. If the fluid parcels could "fall" adiabatically towards their equilibrium position, all the stored potential energy could return back to kinetic energy. However, the damping of temperature fluctuations provokes an irreversible loss of kinetic energy. A simple example of this process is the damping of gravity waves.

Both effects of the thermal diffusivity are thus opposed. While a decrease of the buoyancy force amplitude reduces the associated work and thus the amount of kinetic energy extracted, the second effect increases the fraction of the kinetic energy which is irreversibly lost. Then, for a given mechanical forcing, a relevant question

is whether a larger thermal diffusivity reduces or enhances the kinetic energy of the flow.

We are lacking quantitative results especially in non-linear regimes to answer such a basic question and more generally to understand the effect of thermal diffusivity in a stellar context. This situation is partly due to the difficulty to reproduce flows with realistic Prandtl numbers either in laboratory experiments or by numerical simulations. The Prandtl number $P_r = \nu/\kappa$ which compares the kinematic viscosity ν and the thermal diffusivity κ, varies between 10^{-6} and 10^{-9} within the sun whereas it is equal to 0.7 in the air. Although some fluid like metal liquid may have small Prandtl numbers in laboratory conditions ($P_r = 0.025$ for mercury, see for example Cioni et al. 1997)), these values remains far from the stellar case. The severe numerical limitation is explained by the huge separation between the time scales of viscous dissipation and thermal diffusion. The computation of both processes over a few dynamical times would require a prohibitive amount of computer time.

In this paper, we investigate the limit where the time scale characterizing the thermal exchanges is much shorter than the time scale of the motions (the ratio between both time scales defines the Péclet number). In Sect. 2, an asymptotic form of the governing equations is derived in the context of the Boussinesq approximation. Evidences that these asymptotic equations actually approximate the Boussinesq equations for small Péclet numbers are presented in Sect. 3. Then, in Sect. 4, the elementary properties of the small-Péclet-number equations are described, emphasizing their theoretical and practical interests. Finally, the relevance of this approximation in a stellar context is commented in Sect. 5.

2. Derivation of the small-Péclet-number approximation

We restrict ourselves to a fluid layer embedded in an uniform vertical gravity field and bounded by two horizontal plates. A mechanical forcing is assumed to drive motions which can be described by the Boussinesq approximation. We do not need to specify the forcing for the moment, we only assume that it introduces a velocity scale U_*. The temperature is fixed on both plates so that a linear diffusive profile denoted $T^i(z)$ is established initially. The dynamical effect of the stable stratification is measured by the Brunt-Väisälä frequency, $N_* = (\beta g \Delta T_*/L_*)^{1/2}$, where g denotes the gravity acceleration, β is the thermal expansion coefficient, ΔT_* the temperature difference between the upper and lower plates and L_* the distance separating the plates.

In the context of the Boussinesq approximation, the governing non-dimensional equations read:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + R_i \theta \mathbf{e}_z + \frac{1}{R_e}\nabla^2\mathbf{u}, \tag{1}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta + w = \frac{1}{P_e}\nabla^2\theta, \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3}$$

where, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ is the velocity vector, p the pressure and $\theta(x,y,z) = T(x,y,z) - T^i(z)$ the temperature deviation from the initial temperature profile. The z axis refers to the vertical direction, while the x and y axis refer to the horizontal ones. In the heat equation, the third term of the left hand side corresponds to the vertical advection of temperature against the mean temperature gradient $dT^i(z)/dz$. This gradient is equal to unity in the dimensionless unit. To non-dimensionalize the equations we used the velocity scale U_*, the length scale L_*, the dynamical time scale $t_{\mathrm{D}} = L_*/U_*$, the pressure scale $\varrho_0 U_*^2$ and the temperature variation ΔT_*.

The system is then governed by the Richardson number, R_i, the Péclet number, P_e, and the Reynolds number, R_e, respectively defined as

$$R_i = \left(\frac{N_* L_*}{U_*}\right)^2, \quad P_e = \frac{U_* L_*}{\kappa}, \quad R_e = \frac{U_* L_*}{\nu}.$$

The Richardson number is the square of the ratio between the dynamical time scale t_{D} and the buoyancy time scale $t_{\mathrm{B}} = 1/N_*$. The thermal diffusivity κ appears in the Péclet number which compares the thermal diffusion time scale, $t_\kappa = L_*^2/\kappa$ with the dynamical time scale. The Reynolds number is the ratio between the viscous time scale L_*^2/ν and the dynamical time scale.

In the limit of small Péclet number, we assume that the solutions $\mathbf{u}$ and θ of the Boussinesq equations behave like Taylor series:

$$\mathbf{u} = \mathbf{u_0} + P_e\mathbf{u_1} + P_e^2\mathbf{u_2} + ... \tag{4}$$

$$\theta = \theta_0 + P_e\theta_1 + P_e^2\theta_2 + ... \tag{5}$$

Note that in the context of the Boussinesq equation, the pressure is an intermediate variable determined by the incompressibility condition (3). By inserting these asymptotic expansions in the heat equation, we find at the zero order in P_e:

$$\nabla^2\theta_0 = 0. \tag{6}$$

Since the temperature remains fixed to its initial value on both bounding plates, temperature deviations vanish on both plates. Then, Eq. (6) implies

$$\theta_0 = 0. \tag{7}$$

Thus, at the lowest order in P_e, the Boussinesq equations reduce to the Navier-Stokes equation:

$$\frac{\partial \mathbf{u_0}}{\partial t} + \mathbf{u_0}\cdot\nabla\mathbf{u_0} = -\nabla p_0 + \frac{1}{R_e}\nabla^2\mathbf{u_0}, \tag{8}$$

together with the incompressibility condition,

$$\nabla \cdot \mathbf{u_0} = 0. \tag{9}$$

At this order, the dynamical and thermal equations are decoupled. The coupling is recovered at the first order in P_e:

$$\frac{\partial \mathbf{u_1}}{\partial t} + \mathbf{u_0} \cdot \nabla \mathbf{u_1} + \mathbf{u_1} \cdot \nabla \mathbf{u_0} = -\nabla p_1 + R_i \theta_1 \mathbf{e}_z + \frac{1}{R_e} \nabla^2 \mathbf{u_1}, \tag{10}$$

$$w_0 = \nabla^2 \theta_1, \tag{11}$$

$$\nabla \cdot \mathbf{u_1} = 0. \tag{12}$$

Solutions $\hat{\mathbf{u}} = \mathbf{u_0} + P_e \mathbf{u_1}, \hat{\theta} = \theta_0 + P_e \theta_1$ valid up to the first order in P_e must satisfy the above system of equations (7), (8), (9), (10), (11), (12).

We note that the Lagrangian derivative of temperature deviations does not appear in the heat equation of this system. Thus, at the first order in P_e, one would have found the same system of equations for $\mathbf{u_0}$, $\mathbf{u_1}$, θ_0, θ_1 if the Taylor series had been introduced in the following equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + R_i \theta \mathbf{e}_z + \frac{1}{R_e} \nabla^2 \mathbf{u}, \tag{13}$$

$$P_e w = \nabla^2 \theta, \tag{14}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{15}$$

Therefore, if $\mathbf{u}$ and θ actually behave as Taylor series for small Péclet numbers, the solution of the above equations is identical to the solution of the Boussinesq equations up to the first order in P_e.

The unique difference with the Boussinesq equations comes from the heat equation. Physically, the process leading to the balance $P_e w = \nabla^2 \theta$ can be described as follows: For large values of the thermal diffusivity, the temperature fluctuations are expected to be small and the mean temperature stratification to remain unchanged by the mechanical heat flux. However, vertical motions advecting fluid parcels against the mean temperature gradient always produce temperature deviations and, unlike the nonlinear advection term $\mathbf{u}.\nabla\theta$, this generation process does not depend on the amplitude of the temperature deviations. As fluid parcels go up (or down) in a mean temperature gradient, the amplitude of the temperature deviations tends to increase continuously. In the mean time, thermal diffusion tends to reduce these temperature deviations. Inspection of the heat Eq. (2) shows that this diffusive process can lead to a stationary solution, namely $P_e w = \nabla^2 \theta$. Clearly, if the time scale of the vertical motions is very slow compared to the diffusive time scale, one expects that this stationary solution is practically instantaneously reached. Again, it describes a balance between thermal diffusion and vertical advection against the mean temperature stratification.

In the remainder of this paper, we will refer to the set of equations (13), (14), (15), as the small-Péclet-number equations or as the small-Péclet-number approximation. However, formal mathematical proof that $\mathbf{u}$ and θ actually behave as Taylor series does not exist in the general case. Then, to prove that the small-Péclet-number equations actually approximate the Boussinesq equations in the limit of small Péclet number, specific cases have to be considered. In the next section, we shall present two types of linear flows where the validity of the small-Péclet-number approximation can be proved. Some evidences will also be given for a non-linear flow. The theoretical and practical interest of the small-Péclet-number equations will be emphasized in Sect. 4.

3. Validity of the small-Péclet-number approximation

The first example we consider is that of small amplitude perturbations in a linearly stably stratified atmosphere. The perturbations are resolved into modes proportional to $\exp(-\sigma t)\exp[i(k_x x + k_y y + k_z z)]$ where σ is a complex number and k_x, k_y, k_z, represent the horizontal and vertical wave numbers of the perturbation. In the following, the dispersion relation obtained using the Boussinesq equations is compared to that derived from the small-Péclet-number equations.

The calculation is conducted for two dimensional disturbances ($k_y = 0$), but the three-dimensional case can be readily recovered replacing k_x^2 by $k_x^2 + k_y^2$ in the following expressions. To simplify the presentation we also limit ourselves to the inviscid case. It has been verified that our conclusions are not affected by taking into account the viscosity.

Using the Boussinesq equations, the dispersion relation is:

$$\sigma^2 - \sigma_T \sigma + \sigma_B^2 = 0 \tag{16}$$

whereas the dispersion relation reduces to

$$\sigma = \frac{\sigma_B^2}{\sigma_T} \tag{17}$$

in the context of the small-Péclet-number equations. In these expressions,

$$\sigma_T = \frac{k_x^2 + k_z^2}{P_e}$$

is the damping rate associated with a pure thermal diffusion, and

$$\sigma_B = \sqrt{R_i} \frac{k_x}{\sqrt{k_x^2 + k_z^2}}$$

is the frequency of gravity waves in absence of diffusive processes.

We observe that, in the context of the small-Péclet-number approximation, all disturbances are damped with

a rate equal to σ_B^2/σ_T. On the contrary, the dispersion relation of the Boussinesq equations shows different types of solutions. These solutions are now analyzed for increasing values of the thermal diffusivity.

At small thermal diffusivity, solutions of the dispersion relation correspond to gravity waves damped by thermal diffusion. The two roots of Eq. (16) correspond to two gravity waves propagating in opposed direction. Increasing the thermal diffusion reduces the wave frequency until the roots of (16) become purely real and the associated modes damped without propagating. This occurs when

$$P_e < \frac{\left(k_x^2 + k_z^2\right)^{3/2}}{2\sqrt{R_i}k_x}.$$

It is important to note that, as long as k_z is not equal to zero, there always exists a Péclet number such that this expression is verified for all k_x and k_z. If this was not the case, one could have gravity waves whatever the value of P_e. Then, the small-Péclet-number approximation would not be valid since gravity waves are absent in this approximation.

In deriving the small-Péclet-number equations, we restricted ourselves to a fluid layer bounded vertically. Vertical wave numbers have therefore a lower limit, k_z^{min}, so that all modes are damped without propagation if $P_e < 3\sqrt{3}k_z^{min}/4\sqrt{R_i}$.

Then, the two distinct roots of the dispersion relation correspond to two damping modes. By further increasing the diffusivity, the damping rates take increasingly different values and the associated modes correspond to two different types of motions.

In the limit of small Péclet numbers, the damping rate of the first type of mode is:

$$\sigma = \frac{\sigma_B^2}{\sigma_T} = R_i P_e \frac{k_x^2}{\left(k_x^2 + k_z^2\right)^2}$$

These are exactly the weakly damped modes found in the context of the small-Péclet-number approximation. Note that, despite the high thermal diffusivity, temperature perturbations can be weakly damped, if they are associated with vertical motions against the mean temperature gradient.

The damping rate of the second type of mode is:

$$\sigma = \sigma_T = \frac{k_x^2 + k_z^2}{P_e}$$

Such modes are not found in the context of the small-Péclet-number equations. Note that this is not surprising since they correspond to solutions of the Boussinesq equations which do not behave like Taylor series (see equations (4) and (5)) when the Péclet number goes to zero. These modes undergo a purely diffusive damping which can be made arbitrarily fast as the Péclet number vanishes. Indeed, whatever the values of k_x and k_z, all these modes are reduced by an arbitrary large factor after a time proportional to P_e/k_z^{min}. For this type of motions, the vertical advection term appearing in the linearized heat Eq. (2) is negligible. This shows that, in the limit of small Péclet number, temperature perturbations which are not produced by vertical advection are damped in a very short time.

According to the above discussion, it is always possible to find a Péclet number such that, after an arbitrarily small time, the evolution of the infinitesimal perturbations is equally described by the Boussinesq equations or by the small-Péclet-number equations.

We now consider another example of flow, yet in a linear regime. It concerns the evolution of small disturbances in a stably stratified shear layer. This configuration differs from the previous example by the presence of a mean horizontal flow sheared in the vertical direction. In this case, the validity of the linear version of the small-Péclet-number approximation has already been proved by Dudis' theoretical work (1974). This author considered specifically an hyperbolic-tangent velocity profile in a stable atmosphere characterized by a hyperbolic-tangent temperature profile and used a normal mode approach to study the stability of the flow.

He first determined the neutral stability curve, i.e. the curve separating the stable and unstable regions in the parameter space, for decreasing values of the Péclet number. Then, he showed that for small Péclet numbers these neutral curves could be recovered using a linear version of the small-Péclet-number equations. The convergence of the Boussinesq equations towards the small-Péclet-number equations appears fairly rapid in this case since, already at $P_e = 0.2$, the maximum difference between the neutral curves is within 3 percent. We recently revisited the work of Dudis by considering a linear temperature profile instead of the tangent hyperbolic profile to characterize the stable stratification (Lignières et al. 1999). We confirmed the validity of the small-Péclet-number equations to describe the neutral curves. In addition, we verified its validity for other types of mode (unstable modes symmetric to the shear layer mid-plane) as well as in the viscous case. Note that very rapidly damped modes corresponding to the second type of mode found in the previous discussion may also exist in this case. However, they can not affect the stability of the shear layer since they are very strongly damped.

The third example is a two-dimensional non-linear flow where a shear layer is forced at the top of a linearly stratified fluid. This flow has been studied numerically by Lignières et al. (1998) for large Reynolds numbers ($R_e \approx 2000$ where R_e is based on the layer thickness and the velocity difference across it). Figure 1a shows the typical vorticity field resulting from the destabilization of the shear layer and the concentration of vorticity into vortices. The mean shear and the thermal stratification are

also represented in Figs. 1b and 1c (here, the means refer to horizontal averages).

The other parameters being held fixed, we reduced the Péclet number from 1000 to 1 (equivalently the Prandtl number P_r has been decreased from 0.5 to 5×10^{-4} which already requires some computational effort). Figures 1d and 1e present horizontal profiles of the vertical velocity and the temperature deviation for the two extreme values of the Péclet number. When this number is equal to 1000 (Fig. 1d), one recovers a classical property of the inflexional shear layer instability in stably stratified medium, namely that the phase lag between the vertical velocity and the temperature deviation is $\pi/2$. By contrast, we observe that both fields are antiphased when the Péclet number is equal to unity (Fig. 1e). This striking difference reveals a change in the predominant terms of the heat equation and we verified that this equation is now dominated by a balance between the vertical advection against the mean temperature gradient and the thermal diffusion. These first results are consistent with the convergence of the Boussinesq equation towards the small-Péclet-number approximation. A detailed comparison of the results of these simulations with those obtained by using the asymptotic equations will be reported in a forthcoming paper.

In this section, the validity of the small-Péclet-number approximation has been proved for two types of linear flows. For the case of infinitesimal perturbations in a linearly stably stratified atmosphere, we noted that vertical length scales of perturbations have to be limited to finite values to ensure uniform convergence. Moreover, the Boussinesq and the small-Péclet-number equations give the same solution after an arbitrarily small time, once initial temperature perturbations not associated with vertical advection against the mean temperature gradient have been damped. The example of a non-linear flow we considered is also consistent with the validity of the approximation.

4. Elementary properties of the small-Péclet-number approximation

The elementary properties of the small-Péclet-number equations are analyzed in this section. From a practical point of view, the main interest of these equations is that their numerical integration does not require the computation of the very rapid temporal variation of temperature due to thermal diffusion. The Lagrangian derivative is indeed absent from the asymptotic heat Eq. (14). This property is crucial for the investigation of small Péclet number regimes since numerical simulations are no longer limited by the huge separation between the dynamical and diffusive time scales.

Another simplifying property appears when the Péclet-number equations (13), (14), (15), are written in terms of

$$\psi = \frac{\theta}{P_e}.$$

Using this rescaled temperature deviation, the small-Péclet-number equations become:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + R\psi \mathbf{e}_z + \frac{1}{R_e}\nabla^2 \mathbf{u}, \tag{18}$$

$$w = \nabla^2 \psi, \tag{19}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{20}$$

where

$$R = R_i P_e = \frac{t_{\mathrm{D}} t_\kappa}{{t_{\mathrm{B}}}^2}. \tag{21}$$

Although these equations have been derived as a first order approximation of the Boussinesq equations in the limit of small Péclet number (see Sect. 2), they can also be interpreted as a zero order approximation of the Boussinesq equations in the limit of small Péclet number provided the non-dimensional number $R = R_i P_e$ is assumed to remain finite. Starting from the dimensional Boussinesq equations, one only has to use $P_e \Delta T_*$ as a reference temperature instead of ΔT_* and to assume Taylor-like expansions of the form (4) and (5). Then, provided R remains constant, the above equations arise at the zero order in P_e. The derivation presented in Sect. 2 has been preferred because it does not require the assumption of an infinite Richardson number R_i.

The main interest of the system (18), (19), (20), is that it only depends on two non-dimensional numbers, R and R_e. This is an important simplification as compared to the original Boussinesq equations which are governed by three non-dimensional numbers, R_i, P_e and R_e. This simplification corresponds to the fact that the amplitude of the buoyancy force is no longer determined by two distinct processes, namely the vertical advection against the stable stratification which produces temperature deviations and the thermal diffusion which smoothes them out. It is now determined by a single physical process which combines the effects of both processes.. We shall show below that this process is purely dissipative and that this dissipation is anisotropic (not effective for horizontal motions) and faster for large scale motions.

To do so, we write down the kinetic energy conservation. Multiplying the momentum Eq. (18) by the velocity vector and integrating over the whole domain, we obtain:

$$\frac{dE_{\mathrm{kin}}}{dt} = R\int_V w\psi \, dV - \frac{1}{R_e}\int_V \epsilon \, dV + \int_S \mathbf{F}_{\mathrm{kin}} \cdot \mathbf{dS}. \tag{22}$$

where, on the r.h.s of this equation, the first term is the work done by the buoyancy force, the second term represents the viscous dissipation into heat and the third term is the kinetic energy flux on the surface bounding the domain. Using Eq. (19), the work done by the buoyancy force can be divided into two terms to give:

$$R\int_V w\psi \, dV = -R\int_V (\nabla\psi)^2 \, dV + R\int_S \psi \nabla\psi \cdot \mathbf{dS}. \tag{23}$$

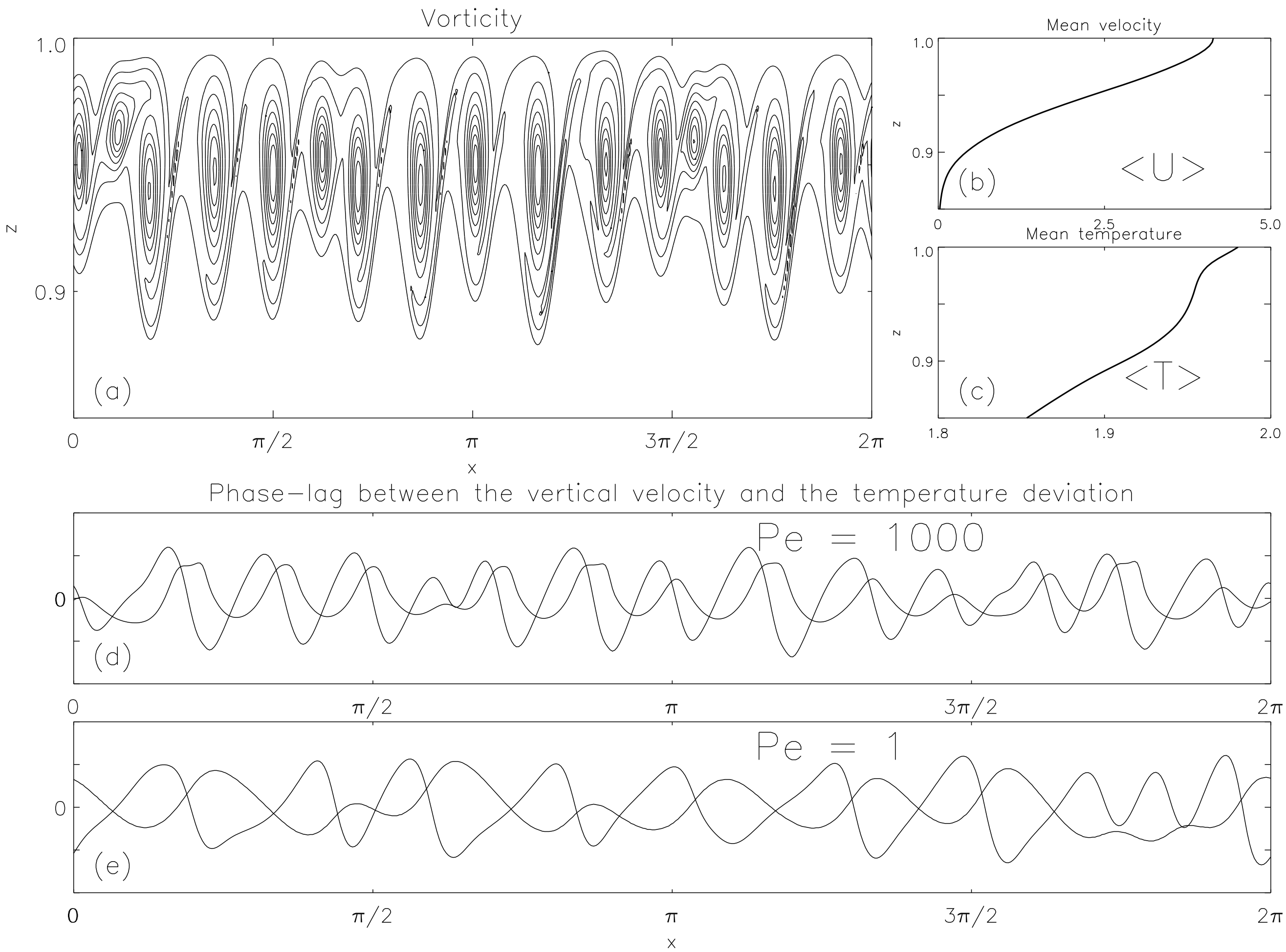

Fig. 1. Effect of the Péclet number on a non-linear two-dimensional shear flow. The figures (a), (b), (c), show a typical vorticity field of the flow and the vertical profiles of the mean horizontal velocity and of the mean temperature, respectively. Horizontal variations of the vertical velocity and of the temperature deviation are displayed for two values of the Péclet number, $P_e = 1000$ (d) and $P_e = 1$ (e). Both quantities are antiphased in the low Péclet number case, this result pointing towards the validity of the small-Péclet-number approximation

As temperature deviations vanish on the bounding plates, the second term also vanishes so that the kinetic energy conservation reduces to:

$$\frac{dE_{\rm kin}}{dt} = -R\int_V (\nabla\psi)^2\ dV - \frac{1}{R_e}\int_V \epsilon\ dV + \int_S \mathbf{F}_{\rm kin}\cdot\mathbf{dS}. \quad (24)$$

This equation shows that the combined effect of the stable stratification and the thermal diffusivity is purely dissipative. This simple result has to be compared with the case of the Boussinesq equations where the integrated work of the buoyancy could be positive or negative. As described in detail by Winters et al. (1995), it is then necessary to distinguish the amount of kinetic energy which is irreversibly lost from the amount of kinetic energy which has been transformed into potential energy but can still return back to kinetic energy. Here, the situation is simpler since all the kinetic energy extracted by the buoyancy work is irreversibly lost.

In order to specify the time scale of this dissipative process, we rewrite the above equations without the non-linear terms and for inviscid motions restricted to a vertical plane $(\mathbf{e}_x, \mathbf{e}_z)$. The pressure term can first be eliminated using the incompressibility condition (20). Then, the two momentum equations are combined to eliminate the horizontal velocity and the simplified heat equation allows to eliminate the rescaled temperature deviation. Finally, the evolution of the vertical velocity is governed by:

$$\frac{\partial \Delta\Delta w}{\partial t} = R\frac{\partial^2 w}{\partial x^2}. \quad (25)$$

Considering isotropic motions of length scale l, the time scale of this process is $1/(Rl^2)$, that is $t_{\rm B}^2/t_\kappa$ in dimensional units. It appears that this dissipative process is faster at large scales than at small scales, which is just the opposite of what is observed in usual dissipative processes like thermal diffusion or viscous dissipation. Here,

however, the thermal diffusion does not act directly on the dynamics; it affects the temperature deviations which in turn modifies the buoyancy force amplitude. We have already seen that, in the limit of small Péclet numbers, rapid thermal exchanges lead instantaneously to a balance between vertical advection against the mean stratification and thermal diffusion. This balance is described by equation (19) and it is straightforward to show that the resulting amplitude of the temperature deviations is stronger if the vertical velocity varies over a large length scale. The amplitude buoyancy force is therefore stronger for velocity fields varying over large length scales and this explains why the combined effect of the stable stratification and the thermal diffusion is faster at larger length scale. Note that, while classical dissipative processes are characterized by a Laplacian operator, the operator of the present dissipation is the inverse of a Laplacian. This can be seen by expressing ψ as the inverse Laplacian of the vertical velocity and by reporting this expression in the momentum equation.

Another interesting property is the anisotropy of this dissipative process. If one considers a velocity field of the form $w \propto \exp[i(k_x x + k_z z)]$, where, as before, k_z and k_x represent its vertical and the horizontal scales, the characteristic time deduced from Eq. (25) is:

$$\tau = \frac{\left(k_x^2 + k_z^2\right)^2}{R k_x^2}, \tag{26}$$

which unsurprisingly corresponds to the inverse the damping rate found in Sect. 2. The use of polar coordinates in the Fourier space is more appropriate to study the anisotropy of the process. With $r^2 = k_x^2 + k_z^2$ and $tan(\alpha) = k_z/k_x$, the time scale becomes:

$$\tau = \frac{1}{R}\left(\frac{r}{\cos(\alpha)}\right)^2 \tag{27}$$

where $\alpha = \pi/2$ corresponds to horizontal motions and $\alpha = 0$ corresponds to vertical motions.

We observe that the dissipation acts primarily on vertical motions while purely horizontal motions are not affected. This is not surprising since the buoyancy force only applies on the vertical component of the velocity. What is more interesting is that this anisotropy is stronger than in the context of the non-diffusive Boussinesq equations. Indeed, for a given value of the wave vector modulus, the time scale τ increases faster towards horizontal motions ($\alpha \to \pi/2$) than the corresponding time scale of the buoyancy force in a non-diffusive atmosphere $1/\sigma_B = 1/(\cos(\alpha)\sqrt{R_i})$. Considering motions strongly affected by the buoyancy force, we thus expect that these motions would be more predominantly horizontal in an atmosphere dominated by thermal diffusion than in a non-diffusive atmosphere.

5. Discussion

In this paper, we derived a small-Péclet-number approximation, we discussed its validity for three flow examples and we analyzed its basic properties. In particular, we showed that the practical and theoretical difficulties characterizing the regime of very large thermal diffusivities and which had been mentioned in the introduction are considerably simplified in the context of the small-Péclet-number approximation.

In what concerns applications for the dynamics of stellar radiative zones, it must be stressed that some type of motions can not be investigated using this approximation. First, there are no gravity waves in the context of the approximation whereas these waves could play an important role in the radiative zone dynamics (Schatzman 1996). Second, thermal convective motions penetrating the radiative zone boundary have a high Péclet number so that the small-Péclet-approximation is not suitable to investigate the overshooting layer at the boundary with the thermal convective zone.

By contrast, there are various evidences that some motions contributing to the radial transport of chemical elements and angular momentum are characterized by very small Péclet numbers and could therefore be studied in the context of the small-Péclet-number approximation. The fact that the thermal structure is determined by the radiative heat flux only shows that the Péclet number characterizing eventual radial motions is necessarily smaller than unity. But other observational constraints, obtained by measuring the surface abundance of chemical elements, give much smaller Péclet numbers (see a recent review by Michaud & Zahn 1998). These Péclet numbers are defined as the ratio between the diffusion coefficient necessary to recover the observed surface abundance and the thermal diffusivity. In the absence of more sophisticated models, this diffusion coefficient is assumed to represent a vertical turbulent transport. For the sun, a Péclet number as small as 2×10^{-4} is obtained. This value may however be underestimated because the turbulence is most probably anisotropic. This aspect is taken into account in the model of Spiegel & Zahn (1992) which describes the tachocline (the abrupt change in angular velocity at the top the solar radiative zone). The Péclet number which characterizes the horizontal turbulent motions and which is compatible with observed thickness of the tachocline remains much smaller than unity ($P_e \approx 10^{-2}$).

These estimates suggest to use the small-Péclet-number approximation to investigate the property of small scale turbulent motions in stellar radiative zones. A first possible investigation could be the homogeneous turbulence in presence of a uniform mean shear and mean temperature gradient. With geophysical applications in mind, this configuration has already been extensively studied for large Péclet numbers (see Schumann 1996, for a review). A comparison with the small-Péclet-number case should

be very instructive. Another important topic concerns the anisotropy between vertical and horizontal motions in an atmosphere dominated by thermal diffusion. Our linear study suggests that this anisotropy can be stronger than in a non-diffusive atmosphere. The ratio between vertical and horizontal turbulent viscosities could be affected and this can estimated through numerical simulations of the small-Péclet-number equations.

Before concluding, it must be noted that the limit of large diffusivities has already been considered (Spiegel 1962, Thual 1992) in the context of the Rayleigh-Bénard convection. However, an important physical property of thermal convection is lost in this limit. The thermal stratification is indeed assumed unchanged and this is not compatible with the general observation that convective motions transform the initial unstable stratification into an adiabatic stratification. There is no such an inconsistency for the case of stably stratified radiative zones we considered here.

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